Slip pulses at a sheared frictional viscoelastic/nondeformable interface

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We study the possibility for a semi-infinite block of linear viscoelastic material, in homogeneous frictional contact with a nondeformable one, to slide under shear via a periodic set of ''self-healing pulses,'' i.e., a set of drifting slip regions separated by stick ones. We show that, contrary to existing experimental indications, such a mode of frictional sliding is impossible for an interface obeying a simple local Coulomb law of solid friction. We then discuss possible physical improvements of the friction model which might open the possibility of such dynamics, among which slip weakening of the friction coefficient, and stress the interest of developing systematic experimental investigations of this question.

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I. INTRODUCTION

A few recent qualitative observations [1,2] on the frictional motion of sheared gels sliding along smooth glass surfaces point towards the existence of inhomogeneous modes of frictional sliding. Namely, in some limited range of values of small shearing rates, sliding seems to occur via propagation of a quasiperiodic pattern of sliding zones of finite width, separated by nonmoving regions, where the interface sticks. These ''slip pulses'' drift at velocities $c \simeq$ mm/s, while the remote average (pulling) velocity lies in the 1–10 μm/s range. Their width is typically tens of micrometers. Analogous observations have been made by Anooshehpoor and Brune [3] on a sliding rubber foam, and by Mouwakeh, Villechaise, and Godet [4] on the elastomer polyurethane.

The topology of such sliding modes is reminiscent of that of Schallamach waves [5], which have been documented [6] in the case of some very compliant transparent rubbers sliding on smooth glass. They consist of quasiperiodic zones, of width typically $l \simeq 100$ μm, with space periods roughly $\sim 10l$ [7], where the rubber buckles, so that the two surfaces separate by a distance comparable with l. These separation waves have drift velocities $\sim$mm/s, for remote velocities $\sim \mu$m/s.

However, the slip pulses in gels do not seem to be associated with any interface separation. In this respect, they are more comparable with the so-called ''self-healing slip pulses,'' on which the attention of mechanicians has been focusing recently [8], following the suggestion by Heaton [9] that some major seismic events may have occurred, not by quasisimultaneous sliding of the whole rupture zone, but via fast propagation of localized sliding zones of small extent.

These observations all point toward a common question about the nature of frictional sliding, which can be schematized as follows. Consider two very thick blocks of solid materials with dissimilar elastic properties, in frictional contact along a planar interface of infinite lateral extent (Fig. 1). This system bears a remote homogeneous compressive stress τ^*_{22}, normal to the interface. Assume that, under the remote shear stress τ^*_{12}, the upper block (I) slides toward $x_1 > 0$ at a remote point velocity v_0 with respect to the lower one (II). Such motion can of course occur in a homogeneous mode, where stresses are uniform. Along the (homogeneous) interface, the friction law, which we assume to obey the Amontons–Coulomb proportionality between shear and normal loads, imposes that

$$\tau^*_{12} = -f_d(v_0)\tau^*_{22}. \tag{1}$$

If such is the case, given τ^*_{22} and v_0, the remote shear sliding stress is fixed. In the Coulomb approximation, where fine variations of the dynamic friction coefficient are ignored, f_d reduces to a constant.

The question then arises as to whether or not this homogeneous sliding mode is stable with respect to small nonhomogeneous perturbations of the stress and strain fields localized in the surface region. In other words, do deformation waves exist along a sliding frictional interface? If so, are they damped, or amplified, or neutral? This question has been studied extensively, for dissimilar linear elastic materials, with Coulomb friction, by several authors, in particular, Weertmann [10], Adams [11], and Martins, Guimaraes, and Faria [12], whose results are synthetized in a recent article by Ranjith and Rice [8]. They find that, when $\mu_d \neq 0$ and when

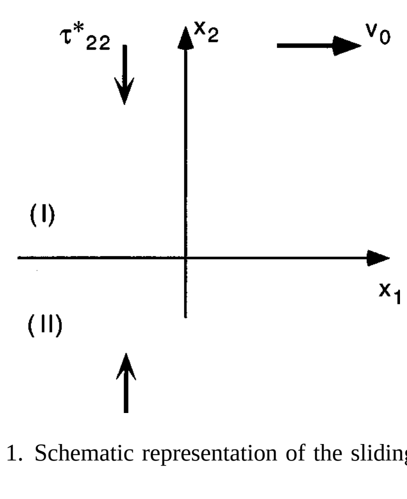

FIG. 1. Schematic representation of the sliding system.

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such interface waves exist [13], the corresponding sliding velocity field along the interface has, for a mode of wavelength k, the form

$$v(x_1)=v_k \exp[ik(x_1-ct)+a|k|t]. \tag{2}$$

Given the elastic moduli, the drift velocity c and amplification coefficient a are real positive constants. That is, waves drifting along (respectively against) the direction of v_0 are amplified (respectively damped). Homogeneous sliding is thus linearly unstable against perturbations of all wavelengths.

These results are derived under the assumption that the interface is sliding everywhere. As amplification proceeds, the sliding velocity necessarily vanishes at some points. This suggests that sliding might occur via a periodic set of self healing slip pulses, separated by stick regions. A family of such pulses has been built by Adams [11] for dissimilar elastic solids. Their drift velocity c, which depends on the values of elastic moduli is, roughly speaking, on the order of a sound velocity. So, their dynamics is controlled by inertia. However, such self-sustaining (stationary) dynamical patterns are singular in the following sense. Since perturbations of all wavelengths, however small, are amplified, any initially localized perturbation gives rise to diverging oscillations at arbitrary small time: Adams's pulses have zero measure attractors. This so-called ''ill-posedness'' most likely signals that the Coulomb friction law misses some of the physical processes which control the fast dynamics of fracture at frictional interfaces between elastic materials, i.e., their high frequency response—a problem which is currently under study [8].

Slip pulses in gels or rubbers, on the contrary, are slow dynamical objects whose velocities, comparable with those of Schallamach waves, are much too low for inertia to be relevant. Their dynamics is certainly controlled by the dissipation associated with the viscoelasticity of these materials.

We therefore concentrate, in this paper, on the following question. Let block (I) be an incompressible linear viscoelastic material with, for simplicity, a single viscous relaxation time. It slides slowly on a smooth nondeformable material, and interface friction obeys a simple local Coulomb law. Under such conditions, are noninertial periodic slip pulses, stationary in the drifting frame, a possible mode of motion?

In Sec. II we formulate the corresponding mathematical problem, and derive the form of its analytical solutions. We show in Sec. III that none of these is compatible with the stick conditions to be satisfied in the nonmoving parts of the interface. Hence, in this as well as in the inertial regime, a Coulomb law with a constant dynamical friction coefficient is incompatible with the existence of such modes of motion. We discuss in Sec. IV possible physical tracks toward improvements of the simple Coulomb model, which might be relevant to the problem of inhomogeneous sliding, and stress the interest of corresponding experimental investigations.

II. GENERAL FORMULATION

We follow closely the approach of Adams [11] and of Comninou and Dundurs [14] restricted to the case where (see Fig. 1) block (II) ($x_2<0$) is nondeformable. Block (I) is submitted to the uniform remote stresses $\tau_{22}^*<0$ and τ_{12}^*. It is infinitely extended along x_1, and made of an incompressible material, with a linear viscoelastic shear response described by

$$\tau_{12}(t)=\int_{-\infty}^{t} dt'\,\mu(t-t')\dot{u}_{12}(t'), \tag{3}$$

where τ_{12} and u_{12} are the shear stress and deformation, confined to the (x_1,x_2) plane, and we model the time dependent shear modulus as a single time Kelvin one, namely,

$$\mu(t)=\mu_\infty+(\mu_0-\mu_\infty)e^{-t/\tau}. \tag{4}$$

We will moreover assume that the relaxed modulus μ_∞ is much smaller than the short time one, μ_0. To fix ideas, for compliant rubbers, values of $\mu_\infty/\mu_0 \lesssim 10^{-3}$ are typical.

We want to study dynamic patterns, where (I) slides toward $x_1>0$ with the uniform remote velocity v_0, with space period $\lambda=2\pi/k$, and drift velocity c in the frame of (II). The corresponding form of the displacements u_1,u_2 reads

$$u_1=v_0 t+\sum_{m\geqslant 1} D_{m1}(x_2)e^{imk(x_1-ct)}, \tag{5}$$

$$u_2=D_{02}+\sum_{m\geqslant 1} D_{m2}(x_2)e^{imk(x_1-ct)}. \tag{6}$$

Solving the wave propagation equation together with the condition of nonseparation at the interface: $u_2(x_1,x_2=0,t)=0$, one obtains straightforwardly (see Appendix A), in the incompressible limit (Poisson coefficient $\nu=1/2$) for the interfacial sliding velocity $v_s(\eta)=\partial u_1/\partial t|_{x_2=0}$

$$v_s(\eta)=v_0+c\,\mathrm{Re}\sum_{m\geqslant 1} B_m e^{im\eta}, \tag{7}$$

where $\eta=k(x_1-ct)$ while the interfacial shear and vertical stresses read

$$\tau_{12}(\eta)=\tau_{12}^*+\mathrm{Re}\sum_{m\geqslant 1} -iB_m\mu_m(1+\sigma_m)e^{im\eta}, \tag{8}$$

$$\tau_{22}(\eta)=\tau_{22}^*+\mathrm{Re}\sum_{m\geqslant 1} B_m\mu_m(1-\sigma_m)e^{im\eta}, \tag{9}$$

with

$$\sigma_m=\left[1-\frac{\rho c^2}{\mu_m}\right]^{1/2} \quad \mathrm{Re}\,\sigma_m>0. \tag{10}$$

We restrict our attention to the very slow modes observed experimentally for which, whatever the frequency (mck), $\rho c^2 \ll \mu_m$, so that

$$\sigma_m \simeq 1-\frac{\rho c^2}{2\mu_m}, \tag{11}$$

where ρ is the mass density of material (I), and μ_m its (complex) elastic modulus at frequency (mck)

$$\mu_m = \hat{\mu}(mck) = \left[-i\omega \int_0^\infty dt \mu(t) e^{i\omega t} \right]_{\omega = mck}. \quad (12)$$

The unknown coefficients B_m must be determined from the second boundary condition along the interface, in which we describe a set of slipping regions of length $2l$ separated by sticking ones.

We assume friction to be described by a simple local Coulomb law, with a constant dynamic friction coefficient f equal to the static one, that is

(i) slip regions: $-\alpha + 2p\pi < \eta < \alpha + 2p\pi$

$$\tau_s(\eta) = \tau_{12}(\eta) + f\tau_{22}(\eta) = 0 \quad v_s > 0, \quad (13)$$

(ii) stick regions: $-\alpha + 2p\pi < \eta < -\alpha + 2(p+1)\pi$

$$v_s = 0 \quad f\tau_{22} < \tau_{12} < -f\tau_{22}. \quad (14)$$

This description of interface friction calls for a few comments. Indeed, it assumes tacitly—as is common in contact mechanics [15]—that one can legitimately define a local and space-independent friction coefficient. Since solid friction results from the average effect of dissipative flips of bistable pinned elastic units, this can be true only on a scale much larger than: (i) the size b of the basic unit, and (ii) the scale L of interface inhomogeneities. The detailed analysis [16] of the Rice–Ruina phenomenological law of dynamic friction [17] has shown that b is of nanometric order. So, our assumption is justified for interfaces with homogeneous intimate contact. Such is indeed the case for the gels or very compliant rubbers which we have in mind here, as long as elastic deformations vary on scales much larger than nanometers—which sets a lower limit on the size of Dugdale–Barenblatt-like fracture head regions.

Note, however, that the situation is different when dealing with multicontact Greenwood-like interfaces [18]. These prevail with stiff materials, such as metals, glasses or rocks, which are not polished down to nanometric roughness. Then, the small scale cutoff is provided by the average distance between contacting asperities, commonly lying in the 100 μm/s range. This, in our opinion, should be kept in mind when attempting, for such interfaces, to regularize the above mentioned ill-posedness problem since, on space scales $\lesssim L$, pinning strength fluctuations become non-negligible.

Taking condition (11) into account, following Comninou and Dundurs [14] we set, for the periodic function $v_s(\eta)$, in $-\pi \leqslant \eta \leqslant \pi$

$$v_s(\eta) = 0 \quad \alpha < |\eta| < \pi \ \ (\text{stick}), \quad (15)$$

$$v_s(\eta) = v(\eta) \quad -\alpha < \eta < \alpha \ \ (\text{slip}). \quad (16)$$

Hence, from Eq. (6)

$$B_m = \frac{1}{\pi c} \int_{-\alpha}^{\alpha} d\xi v(\xi) e^{-im\xi} \quad (m \geqslant 1), \quad (17)$$

$$v_0 = \frac{1}{2\pi} \int_{-\alpha}^{\alpha} d\xi v(\xi). \quad (18)$$

Using Eqs. (15) and (16) together with Eqs. (8) and (9), and with the help of the relation

$$\sum_{m \geqslant 1} e^{imx} = -\frac{1}{2} + \sum_{n=-\infty}^{\infty} \delta(x - 2n\pi) + \frac{i}{2} \mathrm{P}\left[\cot \frac{x}{2} \right], \quad (19)$$

(where P designates the Cauchy principal value), one finally gets, for the interfacial ''sliding stress'' τ_s

$$\tau_s(\eta) = \tau_s^* + \frac{f\rho c^2}{2} [V(\eta) - V_0] - \frac{1}{\pi} \int_{-\alpha}^{\alpha} d\xi V(\xi) ⨍_0^\infty dt \mu(t) \frac{d}{dt} \left[\cot \frac{\eta - \xi + ckt}{2} \right], \quad (20)$$

where we have set $V(\eta) = v(\eta)/c$, and $⨍ f dx \equiv PV \int f dx$.

Integrating by parts the last term on the right hand side (r.h.s.) of Eq. (20), the condition Eq. (13) for frictional sliding within the slip pulses provides us with the integral equation to be satisfied by the interfacial reduced velocity field in $(-\alpha < \eta < \alpha)$, namely,

$$\tau_s^* + \frac{f\rho c^2}{2} [V(\eta) - V_0] + \frac{\mu_0}{\pi} ⨍_{-\alpha}^{\alpha} d\xi V(\xi) \cot \frac{\eta - \xi}{2} + \frac{1}{\pi} \int_{-\alpha}^{\alpha} d\xi V(\xi) ⨍_0^\infty dt \frac{d\mu}{dt} \left[\cot \frac{\eta - \xi + ckt}{2} \right] = 0, \quad (21)$$

with

$$\int_{-\alpha}^{\alpha} d\xi V(\xi) = 2\pi V_0. \quad (22)$$

Once Eqs. (21) and (22) are solved for $V(\eta)$, interfacial stresses in the stick regions should be calculated from Eq. (20), Eq. (14) then providing the final condition for slip pulses to exist.

Expression (21) separates explicitly the instantaneous elastic shear effects (third term) from the contribution of viscoelastic relaxation (fourth term). The second term, which derives from the perturbation of the normal stress τ_{22}, is, for our very slow pulses, smaller than the integral ones by a factor $(c/c_s)^2$, where c_s is some sound velocity. We will therefore neglect it from now on.

The cot form of the elastic kernels results from imposing space periodicity to the patterns. Equation (21) can be rewritten in a form more standard in fracture mechanics by setting in $(-\pi < \eta < \pi)$

$$u = \tan\frac{\eta}{2} \quad y = \tan\frac{\xi}{2} \quad \Phi(u) = \frac{V(u)}{1+u^2} \quad a = \tan\frac{\alpha}{2} = \tan\frac{kl}{2}. \quad (23)$$

Some elementary algebra then leads, in $(-a < u < a)$, to

$$\frac{\tau_s}{1+u^2}=\frac{2\mu_0}{\pi}\int_{-a}^{a}dy\,\frac{\Phi(y)}{u-y}+\int_{-a}^{a}dy\,\Phi(y)k(u,y)-\frac{2V_0u-\tau_s^*}{1+u^2}=0, \quad (24)$$

with

$$\int_{-a}^{a}dy\,\Phi(y)=\pi V_0, \quad (25)$$

and we have set

$$k(u,y)=\frac{2}{\pi}\int_0^\infty dt\mu'(t)\frac{1+y\tan\left(\frac{ckt}{2}\right)}{u-y+(1+uy)\tan\left(\frac{ckt}{2}\right)}. \quad (26)$$

The singular integral equation (24) belongs to a class which was studied extensively by Mushkelishvili [19]. In his terminology, the first two terms on the left hand side (l.h.s.) constitute the ''dominant'' part. The viscoelastic kernel $k(u,y)$ satisfies the regularity condition: $\lim_{u\to y}[(u-y)k(u,y)]=0$. This entails that it plays no role in the strength of the singularities of the solutions, i.e., as is intuitively reasonable, these are ruled by the instantaneous elastic response of the deformable medium.

Following Ref. [19], there are four families of solutions of Eq. (24), each of which is associated with one of the basic functions, characteristic of the dominant part

$$Z_{\epsilon_t,\epsilon_h}(y)=(y+a)^{\epsilon_t/2}(a-y)^{\epsilon_h/2}, \quad (27)$$

where the indices $(\epsilon_{t,h})=(+1,-1)$ control the convergent or divergent behavior of Z at the tail and head edges of the slip zone.

One can then transform, for each family, Eq. (24) into an equivalent nonsingular Fredholm integral equation. It moreover turns out that, when we specialize to the single relaxation time model for $\mu(t)$ [Eq. (4)], analytical expressions for the solutions of these equations can be obtained explicitly, thus allowing us to draw explicit conclusions about their existence.

In view of the heaviness of the (otherwise straightforward) algebra involved, we will exemplify the method in full detail only for one of the families, namely the $(+-)$ one.

III. FOUR FAMILIES OF SOLUTIONS : *V* FIELDS AND EXISTENCE CONDITIONS

A. The $(+-)$ family

The corresponding basic function

$$Z_{+-}(y)=\sqrt{\frac{(y+a)}{(a-y)}}. \quad (28)$$

The implementation of Mushkelishvili's method is performed in Appendix B. For $\mu(t)$ as specified by Eq. (4), the nonsingular equation equivalent to Eq. (24) reads

$$\Phi_{+-}(u)=H_{+-}(u)+\frac{\Delta\mu}{\mu_0}\frac{2\exp(2\tan^{-1}u/ck\tau)}{ck\tau(1+u^2)}\int_u^a dz\Phi_{+-}(z)\times\exp(-2\tan^{-1}z/ck\tau), \quad (29)$$

with

$$H_{+-}(u)=-\frac{Z_{+-}(u)}{2\mu_0}\frac{C^*u+D^*}{1+u^2}+\frac{\Delta\mu}{\mu_0}\frac{W_{+-}}{ck\tau}\left[\frac{Z_{+-}(u)}{\pi}G(u)+\frac{\beta}{1+u^2}\exp(2\tan^{-1}u/ck\tau)\right], \quad (30)$$

$$W_{+-}=2\int_{-a}^{a}dz\Phi(z)\exp\left(-\frac{2\tan^{-1}z}{ck\tau}\right), \quad (31)$$

$$G(u)=-\beta\int_a^\infty d\Psi\frac{e^{2\tan^{-1}\Psi/ck\tau}}{(u-\Psi)(1+\Psi^2)}\sqrt{\frac{\Psi-a}{\Psi+a}}-(1+\beta)\int_{-\infty}^{-a}d\Psi\frac{e^{2\tan^{-1}\Psi/ck\tau}}{(u-\Psi)(1+\Psi^2)}\sqrt{\frac{-\Psi+a}{-\Psi-a}}, \quad (32)$$

$$\beta=(e^{2\pi/ck\tau}-1)^{-1}\quad \Delta\mu=\mu_0-\mu_\infty, \quad (33)$$

$$C^*=2V_0\mu_\infty\cos\frac{\alpha}{2}-\tau_s^*\sin\frac{\alpha}{2}, \quad (34)$$

$$D^*=-2V_0\mu_\infty\sin\frac{\alpha}{2}-\tau_s^*\cos\frac{\alpha}{2}. \quad (35)$$

Expression (32) for $G(u)$ is valid for solutions whose slip zone length $2l$ satisfies the condition $2l<\lambda/2$. We assume this to hold in accordance with experimental observations, which indicate values of $2l/\lambda\ll 1$.

Equation (29) is a first order differential equation for the function $\int_u^a dz\,\Phi_{+-}(z)\exp[-2\tan^{-1}z/ck\tau]$, which is straightforwardly solved in

$$\Phi_{+-}(u)=H_{+-}(u)+\frac{2}{ck\tau}\frac{\Delta\mu}{\mu_0}\frac{1}{1+u^2}\int_u^a dzH_{+-}(z)\times\exp\left[\frac{\mu_\infty}{\mu_0}\frac{2(\tan^{-1}u-\tan^{-1}z)}{ck\tau}\right]. \quad (36)$$

This defines a family of slip velocity fields, each of which is labeled by the four dimensionless parameters $V_0=v_0/c,\tau_s^*/\mu_0,l/\lambda=a/2\pi$, and $c\tau/\lambda$. Two of the physical parameters, v_0 and τ_s^*, are ''external'': in an experiment, one imposes in general an average sliding velocity—hence v_0 is fixed—and measures τ_s^*. l, c, and λ are the internal parameters of the family. This defines a problem of dynamical selection, namely if sliding patterns exist, are l, c, λ, and hence τ_s, uniquely defined when v_0 is fixed, or not? In order to clear up this important question, it is necessary to list the relations between them, or alternately, the conditions to be satisfied by Φ_{+-} as given by Eq. (36). These are:

(i) Two consistency conditions, expressing that the remote velocity and stresses are simply the $k=0$ components of the corresponding fields. This is expressed by relation (25) and by an analogous equation for τ_s

$$\tau_s^* = \frac{1}{\pi}\int_{-\infty}^{+\infty} du \frac{\tau_s(u)}{1+u^2}, \tag{37}$$

where $\tau_s(u)$ is related to $\Phi_{+-}(u)$ by the first of equalities, Eq. (24).

(ii)The interfacial stress field must also satisfy the stick inequality Eq. (14). One easily determines, with the help of Eq. (24), that a divergence of Φ at an edge $u=\pm a$ of the slip zone results in a diverging τ_s at the corresponding stick zone edge, and therefore in the violation of the stick condition. Z_{+-} [Eq. (28)] diverges at the slip head $u=a$. For $u\rightarrow a$

$$\Phi_{+-}(u) = \frac{Z_{+-}(u)}{2\mu_0}\left[-\frac{C^* a + D^*}{1+a^2} + \frac{2\Delta\mu}{\pi}\frac{W_{+-}}{ck\tau}G(a)\right] + \mathrm{Re}(u) \tag{38}$$

with $\lim_{u\rightarrow a} \mathrm{Re}(u)=0$.

A necessary condition for Φ_{+-} to be acceptable is that the coefficient of Z_{+-} in Eq. (38) vanishes, i.e., using Eqs. (29)–(35)

$$\tau_s^* \cos\frac{\alpha}{2} = -\frac{2}{\pi}\Delta\mu\frac{W_{+-}G(a)}{ck\tau}. \tag{39}$$

So, for solutions of the $(+-)$ class, the five pattern parameters are linked by three relations. That is, for a given v_0, this class of patterns, if they exist, form a one parameter family. We will comment further on this conclusion in Sec. IV.

Let us now come back to the‘‘regularization condition,’’ Eq. (39). From conditions (15) and (16), the interfacial sliding stress τ_s must be nonpositive everywhere. Hence, its u average τ_s^* must be strictly negative.

On the other hand, the v_s and thus the Φ field must, by Eq. (13), be positive everywhere in the slip zone. Then definition (31) entails that $W_{+-}>0$. Finally, using Eq. (32), one gets

$$G(a) = -\int_a^\infty d\Psi \frac{\sinh\left[\frac{2}{ck\tau}\left(\frac{\pi}{2}-\tan^{-1}\Psi\right)\right]}{(1+\Psi^2)\sqrt{\Psi^2-a^2}\, sh\left(\frac{\pi}{ck\tau}\right)} < 0. \tag{40}$$

Therefore, condition (39) can never be satisfied. No solution of type $(+-)$ exists. In other words, viscous relaxation and pulse–pulse interaction effects can never be sufficient to cancel the square-root singularity due to the instantaneous elastic response.

B. The $(-+)$ and $(-)$ families

For the $(-+)$ family, whose Z function diverges at the slip zone tail only, the analysis parallels completely the above one, the Φ_{-+} fields obey a set of equations with exactly the same structure as that of Eqs. (29)–(32), differing only in the detailed algebraic expressions of $G(u)$, C^*, and D^*. The regularization condition analogous to Eq. (39), now to be imposed at $u\rightarrow -a$, is again immediately shown to have no solutions.

Z_{--} diverges at both slip edges, hence two regularization conditions, and one shows similarly that the condition obtained from their difference cannot be satisfied. Note, however, that the counting argument tells us that, if $(--)$ patterns could exist, one only of the five parameters would be free, i.e., there could exist at most one dynamical pattern at a given sliding velocity.

C. The $(++)$ family

As Z_{++} vanishes at both slip zone edges, no regularity condition has to be imposed [20]. $(++)$ solutions, if any, form a two parameter family. The analysis of Appendix B again leads to an expression of Φ_{++} with the same structure as Eqs. (29)–(32). One can then write explicitly the self-consistency equation (25) for V_0. Here we will skip the corresponding tedious but straightforward algebra, and only quote the final form of Eq. (25), which can be written as

$$V_0\left[1+O\left(\frac{\mu_\infty}{\mu_0}\right)\right] = \frac{\tau_s^*}{2\mu_0}\frac{1}{ck\tau}\left(1-\cos\frac{\alpha}{2}\right)^2\cos\frac{\alpha}{2}. \tag{41}$$

For the systems we are interested in, as already mentioned, $\mu_\infty/\mu_0 \ll 1$. Then, again, under the stick restriction which imposes that $\tau_s^*<0$, condition (29) cannot be fulfilled.

IV. DISCUSSION

The above analysis leads us to a strong statement, which seems to contradict existing qualitative observations. Namely, an interface with Coulomb friction between a viscoelastic and a nondeformable material cannot sustain slow sliding via a periodic set of alternating noninertial slip pulses and stick regions.

We believe that the reason for this contradiction must be traced to the fact that the Coulomb model of friction which we have assumed misses some physical elements which probably play a crucial role in the dynamics of patterns with fracture-like singularities. This may appear more clearly when one notices that this model, which describes the interface as infinitely rigid below the friction threshold, then, once this is reached, sliding under constant stress, is the exact two-dimensional 2D equivalent of the Hill model of bulk ideal plasticity [21], well known to generate numerous artifact instabilities due to its highly singular character.

Clearly, the main weakness of the Coulomb model lies in its overschematic description of the transition between stick and slip, i.e., for our patterns, in the details of interface boundary conditions at the edges of a slip zone. In the case of the analogous mode-I problem—the Griffith crack—it is well known that discontinuous boundary conditions (vanishing normal stresses and displacements in, respectively, the

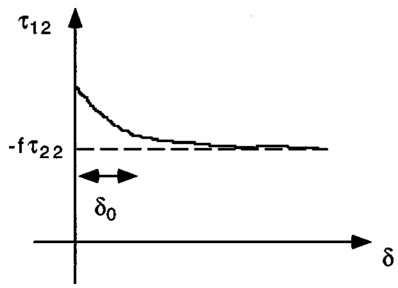

FIG. 2. Schematic representation of the slip-weakening friction law.

cracked and uncracked regions of the crack plane) miss a major physical ingredient, namely the finite range of atomic decohesion and its associated energy cost. Taking this into account regularizes the stress field at the fracture head, by smearing its square root singularity over the Dugdale–Barenblatt cohesive zone [22].

In analogy with this, and based upon the nature of the stress–strain characteristics of overconsolidated clays, Palmer and Rice [23] proposed a model for sliding along a concentrated slip surface in which the sliding shear stress is assumed to decrease with relative displacement as shown in Fig. 2. This enabled them to analyze the ''mode-II fracture'' problem of shear band propagation in such materials.

Let us assume for the moment that we can modify our Coulomb model in a similar manner. That is, let us assume that the shear stress in the sliding state is given by

$$\tau_{12} = -f\tau_{22} + \delta\tau_0[\delta(x_1,t)], \tag{42}$$

where $\delta\tau_0$ is maximum for $\delta=0$ and has a small range $\delta_0 \ll 2l$. We define $\delta(x_1,t)$ as the displacement at the interface point x_1 from its position when it was in the preceding stick zone, i.e., up until the head of the slip zone under consideration reached it. So

$$\begin{aligned}\delta(x_1,t) &= u_1(x_1,t) - u_1\left(x_1, t - \frac{a-x_1}{c}\right)\\ &= \int_\eta^\alpha d\eta' V(\eta')\\ &\equiv \delta(\eta).\end{aligned} \tag{43}$$

In Eq. (21) for the velocity field, τ_s^* must now be substituted by $\tau_s^* - \delta\tau_0[\delta(\eta)]$.

In order to fix ideas, let us concentrate on $(+-)$ solutions. Repeating the analysis of Appendix B leads again to expression (36), with

$$H_{+-}(u) \to H_{+-}(u) - \frac{Z_{+-}(u)}{2\mu_0} I(a), \tag{44}$$

$$I(a) = -\frac{1}{\pi} \int_{-a}^{a} dy\, \frac{\delta\tau_0[\delta(y)]}{(1+y^2)(a-y)Z_{+-}(y)}. \tag{45}$$

The regularity condition then becomes

$$\tau_s^* \cos\frac{\alpha}{2} + \frac{2\Delta\mu}{\pi}\, \frac{W_{+-}G(a)}{ck\tau} = \frac{I(a)}{\pi}. \tag{46}$$

The lhs of Eq. (46) must, as shown above, be negative. From Eq. (45), $I(a)<0$. So, the introduction of a ''cost for incipient sliding'' via $\delta\tau_0$ is sufficient to lift the incompatibility which we found to hold for the Coulomb friction model. Clearly, the same formal result applies for the $(-+)$ and $(--)$ classes. That is, the localized incremental stress $\delta\tau_0$ plays a role comparable to that of the cohesive stress in mode-I fracture, namely it smooths out the stress singularity by spreading it over a zone of incipient sliding of small but finite extent.

However, once this formal remark has been made, one should come back to the possible physical interpretation of such a modification of the friction model. A decrease in the frictional stress with the slip distance is likely to be associated with a change upon sliding of the internal structure of the nanometer-thick adhesive interfacial junction. Moreover, in order for the peaked structure of $\tau_{12}(\delta)$ to reproduce itself at each successive slip zone head, the structure of the junction must relax non-negligibly on the duration $\Delta\tau_{st} = (\lambda - 2l)/c$ of a stick (typically, $\Delta\tau_{st}$ lies in the range of seconds).

Such a scenario is plausible for junctions composed of long molecules—either because a molecular layer of lubricant is present or because the junction is formed by molecular tails from the sliding material itself. Then, sliding is likely to give rise to a slip weakening of friction associated with molecular elongation and restrengthening by structural relaxation during stick. These are precisely the physical ingredients invoked to explain the hysteretic frictional dynamics observed in a number of boundary lubrication experiments [24–26].

However, inclusion of slip weakening of dynamical friction is not the only possible improvement on the Coulomb model susceptible to allow for slip pulses. Indeed, a series of recent works by Langer *et al.* [27,28] on the viscoplasticity of amorphous solids point very convincingly toward the crucial importance of a realistic description—of the rate and state type—of the gradual crossover of the mechanical shear response from mainly elastic to mainly dissipative. As already pointed out, solid friction along a continuous interface is nothing but 2D interfacial viscoplasticity, to which the bulk analysis should be transposable. Hence the need for the elaboration of a phenomenolgy which can bridge realistically between static and dynamic solid friction. Work in that direction, based upon experimental studies of dynamic interfacial shear response, is presently in progress.

This discussion naturally leads us to emphasize the need for the development of systematic experimental studies of interfacial slip pulses, and the interest which they present. The main questions to be elucidated are:

(i) The precise conditions for frictional sliding to occur in this mode. This includes systematic characterization of the bulk viscoelasticity of systems which do exhibit this behavior, and qualification of the relevant range of driving velocities v_0.

(ii) the v_0 dependence of the apparent friction coefficient $\tau_{12}^*/|\tau_{22}^*|$, and the question of pattern selection. Namely, is the slip pattern unique for a given v_0 or, for example, does it depend on the lateral size of the sliding block? In other words, does injection at the back free edge of the slider play a crucial role in the selection of the pattern wavelength λ or not? Further elucidation of these questions would also be of value to shed further light on the still largely open question of the physics of shear interfacial fracture.

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APPENDIX A

We briefly sketch here the derivation of Eqs. (7)–(9) for a purely elastic system. Let λ,μ be its Lame coefficients, related to the Young modulus E and to the Poisson ratio ν by

$$\mu=\frac{E}{2(1+\nu)}\qquad \lambda+\mu=\frac{E}{2(1+\nu)(1-2\nu)}. \tag{A1}$$

The elastic displacement $\mathbf{u}=(u_1,u_2)$ obeys the Lame equation

$$\rho\ddot{\mathbf{u}}=(\lambda+\mu)\boldsymbol{\nabla}\cdot\mathrm{div}\,\mathbf{u}+\mu\Delta\mathbf{u}, \tag{A2}$$

with ρ the mass density. The stresses are given by

$$\frac{\tau_{ij}}{\mu}=\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}+(\beta^2-2)\,\delta_{ik}\mathrm{div}\,\mathbf{u}, \tag{A3}$$

where

$$\beta^2=\frac{2(1-\nu)}{1-2\nu}. \tag{A4}$$

One then sets

$$u_i(x_1,x_2,t)=u_i^*(x_2)+\sum_m U_{im}(x_2)e^{imk(x_1-ct)}, \tag{A5}$$

with $\mathbf{u}^*$ the displacement field corresponding to uniform sliding under the homogenenous stresses τ^*. Solving Eq. (A2) together with the condition of nonseparation at the interface $u_2|_{x_2=0}=0$, one gets

$$u_1(x_1,x_2,t)-u_1^*(x_2)=\mathrm{Re}\sum_{m\geqslant 1}A_m\left[-\frac{k^2}{s_+s_-}e^{-ms_+x_2}+e^{-ms_-x_2}\right]e^{imk(x_1-ct)}, \tag{A6}$$

$$u_2(x_1,x_2,t)-u_2^*=\mathrm{Re}\sum_{m\geqslant 1}A_m\frac{ik}{s_-}[-e^{-ms_+x_2}+e^{-ms_-x_2}]e^{imk(x_1-ct)}, \tag{A7}$$

with

$$s_+^2=k^2\left(1-\frac{\rho c^2}{\lambda+2\mu}\right)\qquad s_-^2=k^2\left(1-\frac{\rho c^2}{\mu}\right)\qquad \mathrm{Re}\,s_\pm>0. \tag{A8}$$

Then, with the help of Eq. (A3), and in the incompressible limit $\lambda\to\infty$, one obtains the expression for the interface stresses and the interface sliding velocity

$$\tau_{12}|_{x_2=0}=\tau_{12}^*+\mu\,\mathrm{Re}\sum_{m\geqslant 1}mkA_m\left(\frac{k}{s_-}-\frac{s_-}{k}\right)e^{imk(x_1-ct)}, \tag{A9}$$

$$\tau_{22}|_{x_2=0}=\tau_{22}^*+\mu\,\mathrm{Re}\sum_{m\geqslant 1}imkA_m\times\left[-2+\frac{k}{s_-}+\frac{s_-}{k}\right]e^{imk(x_1-ct)}, \tag{A10}$$

$$v_s=v_0-c\,\mathrm{Re}\sum_{m\geqslant 1}imkA_m\left[1-\frac{k}{s_-}\right]e^{imk(x_1-ct)}. \tag{A11}$$

Setting

$$B_m=imkA_m\left(\frac{k}{s_-}-1\right) \tag{A12}$$

and substituting s_- by $s_{m-}=k\sigma_m$ [Eq. (A3)] appropriate to the viscoelastic system directly yields expressions (7)–(11).

APPENDIX B

Following [19], the singular integral equation (24), valid in $-a<u<a$

$$\frac{2\mu_0}{\pi}\fint_{-a}^{a}dy\,\frac{\Phi(y)}{u-y}+\int_{-a}^{a}dy\Phi(y)k(u,y)=F(u), \tag{B1}$$

where k is given by Eq. (26) and

$$F(u)=\frac{2V_0u-\tau_s^*}{1+u^2} \tag{B2}$$

is equivalent for the $(+-)$ family of solutions to

$$\Phi+K\star k\star\Phi=K\star F \tag{B3}$$

with

$$[K\star f](u)=-\frac{Z_{+-}(u)}{2\mu_0\pi}\fint_{-a}^{a}dy\,\frac{f(y)}{Z_{+-}(y)(u-y)}. \tag{B4}$$

Integrating in the complex y plane along the contour shown in Fig. 3, one finds

$$K\star F=-\frac{Z_{+-}(u)}{2\mu_0}\,\frac{C^*u+D^*}{1+u^2}, \tag{B5}$$

where C^*,D^* are given by Eqs. (34) and (35).

On the other hand

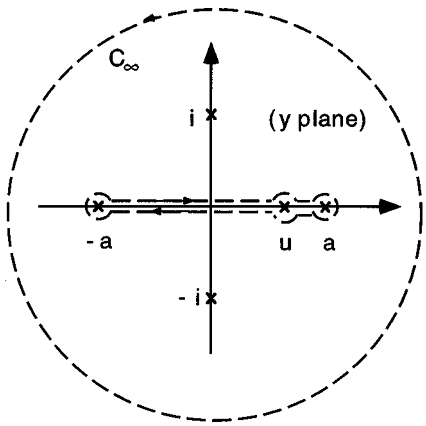

FIG. 3. Contour for integrals of type $K\star F$.

$$(K\star k\star\Phi)(u)=-\frac{Z_{+-}(u)}{2\mu_0\pi}\int_{-a}^{a}dz\,\Phi(z)J_{\rightarrow}(y,z),\quad\text{(B6)}$$

$$J_{\rightarrow}(y,z)=\frac{2}{\pi}\fint_{-a}^{a}\frac{dy}{Z_{+-}(y)(u-y)}\fint_{0}^{\infty}ds\,\frac{\mu'(s)}{y-\dfrac{z-T(s)}{1+zT(s)}},\tag{B7}$$

with

$$T(s)=\tan\frac{cks}{2}.\tag{B8}$$

Once the order of the y and s integrals on the r.h.s. of Eq. (B7) has been interchanged, the y integration can be performed explicitly. However, care must be exercised when performing this interchange, due to the presence of the two principal values. One uses the following identity, which results from the Poincare–Bertrand theorem [19]:

$$\begin{aligned}\mathrm{P}\left(\frac{1}{x''-x'}\right)\mathrm{P}\left(\frac{1}{x''-x}\right)&=\mathrm{P}\left(\frac{1}{x-x'}\right)\left[\mathrm{P}\left(\frac{1}{x''-x}\right)-\mathrm{P}\left(\frac{1}{x''-x'}\right)\right]\\&\quad+\pi^2\delta(x''-x')\,\delta(x''-x).\end{aligned}\tag{B9}$$

One thus obtains

$$J_{\rightarrow}(y,z)=\fint_{0}^{\infty}ds\,\mu'(s)\times\fint_{-a}^{a}\frac{dy}{Z_{+-}(y)(u-y)\left(y-\dfrac{z-T(s)}{1+zT(s)}\right)}+Y,\tag{B10}$$

$$Y=-\pi^2\sum_{p=-\infty}^{\infty}\frac{\mu'(s_p(z,u))}{|\partial D/\partial s|_{s_p}Z_{+-}(u)}\,\theta(s_p(z,u)),\tag{B11}$$

where the s_p are the zeros of $D=u-(z-T(s))/(1+zT(s))$, i.e.,

$$s_p(z,u)=\frac{2}{ck}[\phi(z,u)+p\pi],\tag{B12}$$

$$\phi(z,u)=\tan^{-1}\left[\frac{z-u}{1+zu}\right]\qquad-\frac{\pi}{2}<\phi(z,u)<\frac{\pi}{2}.\tag{B13}$$

From this one finally gets

$$\begin{aligned}(K\star k\star\Phi)(u)=&\frac{2}{\mu_0ck(1+u^2)}\int_{-a}^{a}dz\,\Phi((z)\mu'[s_0(z,u)]\\&\times[\theta(\phi(z,u))+\beta]-\frac{Z_{+-}(u)}{2\mu_0}\\&\times\int_{-a}^{a}dz\,\Phi(z)\fint_{0}^{\infty}ds\,\frac{\mu'(s)\,\theta(\psi^2-a^2)}{Z_{+-}^{\mathrm{out}}(\psi)(u-\psi)},\end{aligned}\tag{B14}$$

where β is defined in Eq. (33) and

$$Z_{+-}^{\mathrm{out}}(\psi)=\theta(\psi-a)\sqrt{\frac{\psi+a}{\psi-a}}+\theta(-\psi-a)\sqrt{\frac{-\psi-a}{-\psi+a}}.\tag{B15}$$

Straightforward integration then results in Eqs. (29)–(32).

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